COHEN-MACAULAY EDGE IDEAL WHOSE HEIGHT IS HALF OF THE NUMBER OF VERTICES

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Abstract. We consider a class of graphs G such that the height of the edge ideal $I(G)$ is half of the number $\sharp V(G)$ of the vertices. We give Cohen-Macaulay criteria for such graphs.

Introduction

In this article a graph means a simple graph without loops and multiple edges. Let G be a graph with the vertex set $V(G) = \{x_1, \ldots, x_n\}$ and with the edge set $E(G)$. Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field K. The *edge ideal* $I(G)$, associated to G, is the ideal of S generated by the set of all squarefree monomials $x_i x_j$ so that x_i is adjacent to x_j. For this ideal the following theorem [5] is known:

Theorem 0.1. *Suppose G is an unmixed graph without isolated vertices. Then we have* $2\,\mathrm{height}\, I(G) \geq \sharp V(G)$.

In this paper we treat the class of graphs for which the above equality holds, i.e., we consider an unmixed graph without isolated vertex with $2\,\mathrm{height}\, I(G) = \sharp V(G)$. Such a class of graphs is rich, because it includes all the unmixed bipartite graphs and all the grafted graphs. Herzog-Hibi [8] gave beautiful theorems on Cohen-Macaulay edge ideals of bipartite graphs. Our purpose in this article is to generalize their results for our class of graphs.

It is known that a graph G in our class has a perfect matching, we may assume that

(*) $V(G) = X \cup Y$, $X \cap Y = \emptyset$, where $X = \{x_1, \ldots, x_n\}$ is a minimal vertex cover of G and $Y = \{y_1, \ldots, y_n\}$ is a maximal independent set of G such that $\{x_1y_1, \ldots, x_ny_n\} \subset E(G)$.

Date: September 24, 2009.

2000 *Mathematics Subject Classification.* Primary 05C75, Secondary 05C90, 13H10, 55U10.

Key words and phrases. Unmixed graph, Cohen-Macaulay graph.

Hence $\{x_1 - y_1, \ldots, x_n - y_n\}$ is a system of parameters of $S/I(G)$. In Sections 3 and 4, using this, we give the following characterization of Cohen-Macaulayness, which is similar to the case of bipartite graph (see [8]).

Theorem 0.2. *Let G be an unmixed graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$. *Then the following conditions are equivalent:*

1. *G is Cohen-Macaulay.*
2. *$\Delta(G)$ is strongly connected.*
3. *There is a unique perfect matching in G.*
4. *$\Delta(G)$ is shellable.*

Note that it includes equivalence between Cohen-Macaulayness and shellability as in the bipartite graphs (see [3]).

We also have a Cohen-Macaulay criterion which is similar to Herzog-Hibi ([8], Theorem 3.4):

Theorem 0.3. *Let G be a graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$. *We assume the conditions (*) and*

$$(**)\ x_i y_j \in E(G) \text{ implies } i \leq j.$$

Then the following conditions are equivalent:

1. *G is Cohen-Macaulay.*
2. *G is unmixed.*
3. *The following conditions hold:*
   (i) *If $z_i x_j, y_j x_k \in E(G)$, then $z_i x_k \in E(G)$ for distinct i, j, k and for $z_i \in \{x_i, y_i\}$.*
   (ii) *If $x_i y_j \in E(G)$, then $x_i x_j \notin E(G)$.*

Although in Herzog-Hibi [8] Alexander duality plays an important role for their proof, we give a direct and elementary proof without it. The Herzog-Hibi criterion for bipartite graphs was discussed by many authors in literature that gave alternative proofs for it (see [7], [13])

In Section 5 we introduce a new class of graphs which we call B-grafted graphs. They are a generalization of grafted graphs introduced by Faridi [4]. If G is an unmixed B-grafted graph, then we have $2\,\text{height}\, I(G) = \sharp V(G)$. Hence applying our main result, we show:

Theorem 0.4. *The B-grafted graph $G(H_0; B_1, \ldots, B_p)$ is Cohen-Macaulay (unmixed, respectively) if and only if every bipartite graph B_i is Cohen-Macaulay (unmixed, respectively) for $i = 1, \ldots, p$.*

See Sections 1 and 5 for undefined concepts and notation.

1. Preliminaries

In this section we recall some concepts and a notation on graphs and on simplicial complexes that we will use in the article.

Let G be a graph with the vertex set $V(G) = \{x_1 \ldots, x_n\}$ and with the edge set $E(G)$. The *induced subgragh* $G|_W$ by $W \subset V(G)$ is defined by

$$G|_W = (W, \{e \in E(G); e \subset W\}).$$

For $W \subset V(G)$ we denote $G|_{V(G)\setminus W}$ by $G - W$. For $F \subset E(G)$ we denote $(V(G), E(G) \setminus F)$ by $G - F$. For a family F of 2-element subsets of $V(G)$ we denote $(V(G), E(G) \cup F)$ by $G + F$.

A subset $C \subset V(G)$ is a *vertex cover* of G if every edge of G is incident with at least one vertex in C. A vertex cover C of G is called *minimal* if there is no proper subset of C which is a vertex cover of G. A subset A of $V(G)$ is called an *independent set* of G if no two vertices of A are adjacent. An independent set A of G is *maximal* if there exists no independent set which properly includes A. Observe that C is a minimal vertex cover of G if and only if $V(G) \setminus C$ is a maximal independent set of G. And also note that height $I(G)$ is equal to the smallest number $\sharp C$ of vertices among all the minimal vertex covers C of G. A graph G is called *unmixed* if all the minimal vertex covers of G have the same number of elements. A graph G is called *Cohen-Macaulay* if $S/I(G)$ is a Cohen-Macaulay ring, where $S = K[x_1, \ldots, x_n]$ is a polynomial ring for a field K. Refer [2], [14] for detailed information on this subject.

Set $V = \{x_1, \ldots, x_n\}$. A *simplicial complex* Δ on the vertex set V is a collection of subsets of V such that (i) $\{x_i\} \in \Delta$ for all $x_i \in V$ and (ii) $F \in \Delta$ and $G \subseteq F$ imply $G \in \Delta$. An element $F \in \Delta$ is called a *face* of Δ. For $F \subset V$ we define the *dimension* of F by $\dim F = \sharp F - 1$, where $\sharp F$ is the cardinality of the set F. A maximal face of Δ with respect to inclusion is called a *facet* of Δ. If all facets of Δ have the same dimension, then Δ is called *pure*.

A pure simplicial complex Δ is called *shellable* if the facets of Δ can be given a linear order $F_1, \ldots, F_m$ such that for all $1 \leq j < i \leq m$, there exist some $v \in F_i \setminus F_j$ and some $k \in \{1, \ldots, i-1\}$ with $F_i \setminus F_k = \{v\}$.

Moreover, a pure simplicial complex Δ is *strongly connected* if for every two facets F and G of Δ there is a sequence of facets $F = F_0, F_1, \ldots, F_m = G$ such that $\dim(F_i \cap F_{i+1}) = \dim \Delta - 1$ for each $i = 0, \ldots, m-1$.

If G is a graph, we define the *complementary simplicial complex* of G by

$$\Delta(G) = \{A \subseteq V(G) : A \text{ is an independent set in } G\}.$$

Observe that $\Delta(G)$ is the Stanley-Reisner simplicial complex of $I(G)$.

2. Unmixedness

In this section we survey unmixed graphs whose edge ideals have the height that is half of the number of vertices.

Lemma 2.1. *Let G be an unmixed graph with non-isolated $2n$ vertices and with* $\operatorname{height} I(G) = n$. *Then G has a perfect matching.*

The proof is clear from ([6], Remark 2.2). By the lemma for an unmixed graph G with $2n$ vertices, which are not isolated, and with $\operatorname{height} I(G) = n$, we may assume

(*) $V(G) = X \cup Y$, $X \cap Y = \emptyset$, where $X = \{x_1, \dots, x_n\}$ is a minimal vertex cover of G and $Y = \{y_1, \dots, y_n\}$ is a maximal independent set of G such that $\{x_1y_1, \dots, x_ny_n\} \subset E(G)$.

From now on, set $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ for a field K and $I(G)$ is an ideal of S. By Lemma 2.1 we have the following ring-theoretic properties of $S/I(G)$:

Corollary 2.2. *Let G be an unmixed graph with $2n$ vertices, which are not isolated, and with* $\operatorname{height} I(G) = n$. *We assume the condition (*). Then*

(i) *Each minimal prime ideal of $I(G)$ is of the form*

$$(x_{i_1}, \dots, x_{i_k}, y_{i_{k+1}}, \dots, y_{i_n}),$$

where $\{i_1, \dots, i_n\} = \{1, \dots, n\}$.

(ii) *$\{x_1 - y_1, \dots, x_n - y_n\}$ is a system of parameters of $S/I(G)$.*

For later use we give a characterization of the unmixedness for our graphs, that is a more detailed description, but a special case of a more general result ([10], Theorem2.9):

Proposition 2.3. *Let G be a graph with $2n$ vertices, which are not isolated, and with* $\operatorname{height} I(G) = n$. *We assume the condition (*). Then G is unmixed if and only if the following conditions hold:*

(i) *If z_ix_j, $y_jx_k \in E(G)$ then $z_ix_k \in E(G)$ for distinct i, j and k and for $z_i \in \{x_i, y_i\}$.*

(ii) *If $x_iy_j \in E(G)$ then $x_ix_j \notin E(G)$.*

3. Cohen-Macaulayness

In this section we give combinatorial characterizations of Cohen-Macaulay graphs whose edge ideals have the height that is half of the number of vertices.

First we introduce an operator that allows us to construct a new graph. Let G be a graph with $2n$ vertices, which are not isolated, and with $\operatorname{height} I(G) = n$. We assume the condition (*).

For any $i \in [n] := \{1, \dots, n\}$ set

$$E_i := \{k \in [n] \,:\, x_ky_i \in E(G)\} \setminus \{i\},$$

and define the graph $O_i(G)$ by

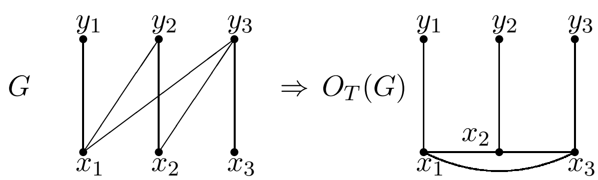
$$O_i(G) := G - \{x_k y_i \,:\, k \in E_i\} + \{x_k x_i \,:\, k \in E_i\}.$$

Then for every subset $T := \{i_1, \dots, i_\ell\}$ of the set $[n]$, we define

$$O_T(G) = O_{i_1} O_{i_2} \cdots O_{i_\ell}(G).$$

Note that $O_T(G)$ is a graph with $2n$ vertices, which are not isolated, and with height $I(G) = n$ satisfying the condition (*).

Example 3.1. *Let $T = \{2, 3\}$, then*



The next proposition shows that Cohen-Macaulayness of G can be checked by unmixedness of all the deformations $O_T(G)$ of G.

Proposition 3.2. *Let G be an unmixed graph with $2n$ vertices, which are not isolated, and* height $I(G) = n$*. We assume the condition (*). Then the following conditions are equivalent:*

(1) *G is Cohen-Macaulay.*
(2) *$O_T(G)$ is Cohen-Macaulay for every subset T of $[n]$.*
(3) *$O_T(G)$ is unmixed for every subset T of $[n]$.*

Proof. Set $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$, $S_k = K[x_1, \dots, x_n, y_{k+1}, \dots, y_n]$, and $G_k = O_{T_k}(G)|_{X \cup \{y_{k+1}, \dots, y_n\}}$.

(1) $\Longrightarrow$ (2). By relabeling, we may assume that $T = [k]$. Let G be a Cohen-Macaulay graph. Then

$$S/(I(G) + (x_1 - y_1, \dots, x_k - y_k)) \simeq S_k/(I(G_k) + (x_1^2, \dots, x_k^2))$$

is Cohen-Macaulay. Since the polarization preserves Cohen-Macaulayness,

$$S/(I(G_k) + (x_1^2, \dots, x_k^2))^{\text{pol}} = S/(I(G_k) + (x_1 y_1, \dots, x_k y_k)) = S/I(O_T(G))$$

is Cohen-Macaulay, where $(x_1^2, \dots, x_k^2)^{\text{pol}}$ stands for the polarization of $(x_1^2, \dots, x_k^2)$. See [12] for basic properties of polarization.

(2) $\Longrightarrow$ (3). Every Cohen-Macaulay ideal is unmixed [1].

(3) $\Longrightarrow$ (1). Suppose G is not Cohen-Macaulay. We want to prove that there exists a subset $T \subset [n]$ such that $O_T(G)$ is not unmixed. Since G is not Cohen-Macaulay the sequence $\{x_i - y_i : 1 \le i \le n\}$ is not a regular sequence of $S/I(G)$. Hence there exists $k \ge 1$ such that $\{x_i - y_i : i \in [k-1]\}$ is a regular sequence of $S/I(G)$ and $x_k - y_k$ is not regular on the ring

$$R := S_{k-1}/(I(G_{k-1}) + (x_1^2, \dots, x_{k-1}^2)) \simeq S/(I(G) + (x_1 - y_1, \dots, x_{k-1} - y_{k-1})).$$

Set $J = I(G_{k-1}) + (x_1^2, \ldots, x_{k-1}^2)$. Since $x_k - y_k$ is not regular on R, then

$$x_k - y_k \in \bigcup_{P \in \operatorname{Ass} R} P$$

and there exists an associated prime ideal $\widetilde{P}$ of J such that $x_k - y_k \in \widetilde{P}$. Since $x_k \in \widetilde{P}$ or $y_k \in \widetilde{P}$, we have $x_k, y_k \in \widetilde{P}$. Hence height $\widetilde{P} > n$. Hence R is not unmixed. Therefore $S/(I(G_{k-1}) + (x_1^2, \ldots, x_{k-1}^2))^{\mathrm{pol}} \simeq S/I(O_{T_{k-1}}(G))$ is not unmixed. □

For distinct $i_1, i_2, \ldots, i_r \in [n]$ we denote by $C_{i_1 i_2 \ldots i_r}$ the cycle C with

$$V(C) = \{x_{i_1}, y_{i_1}, x_{i_2}, \ldots, x_{i_r}, y_{i_r}\}$$

and

$$E(C) = \{x_{i_1} y_{i_1}, y_{i_1} x_{i_2}, x_{i_2} y_{i_2}, \ldots, y_{i_r} x_{i_r}, y_{i_r} x_{i_1}\}.$$

Proposition 3.3. *Let G be an unmixed graph with $2n$ vertices, which are not isolated, and* height $I(G) = n$. *We assume the condition (*). Then the following conditions are equivalent:*

1. *The subset $\{x_1y_1, x_2y_2, \ldots, x_ny_n\}$ of $E(G)$ is a unique perfect matching in G.*
2. *The cycle C_{ij} is not included in G for any $i < j$.*
3. *For any $r \geq 2$ the cycle $C_{i_1 i_2 \ldots i_r}$ is not included in G for any subset $\{i_1, i_2, \ldots, i_r\} \subset [n]$ of cardinality r.*

Proof. (1) $\Longrightarrow$ (2). Suppose $C_{ij} \subset G$. Then we have two perfect matchings in G:

$$\{x_1y_1, x_2y_2, \ldots, x_ny_n\},$$
$$\{x_1y_1, x_2y_2, \ldots, x_{i-1}y_{i-1}, x_iy_j, x_jy_i, x_{i+1}y_{i+1}, \ldots, x_ny_n\}.$$

(2)$\Longrightarrow$ (3). We proceed by induction on r.

For $r = 2$ there is nothing to prove. Assume $r > 2$ and suppose that $C_{i_1 i_2 \ldots i_r} \subset G$. Since $y_{i_{r-1}} x_{i_r}, y_{i_r} x_{i_1} \in E(G)$, we have $y_{i_{r-1}} x_{i_1} \in E(G)$ by Theorem 2.3. Hence $C_{i_1 i_2 \ldots i_{r-1}} \subset G$, which is a contradiction with the inductive hypothesis.

(3) $\Longrightarrow$ (1). Suppose there exists another perfect matching:

$$\{x_1 y_{i_1}, x_2 y_{i_2}, \ldots, x_n y_{i_n}\} \subset E(G).$$

Then we define a permutation σ by

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ i_1 & i_2 & \ldots & i_n \end{pmatrix}.$$

Then σ can be decomposed as $\sigma = \prod \sigma_i$, where each σ_i is a cycle of σ. Since σ is not an identity permutation, for some i the cycle σ_i is of the form $(j_1\, j_2\, \ldots\, j_r)$ with $r \geq 2$. Then we have $C_{j_r j_{r-1} \ldots j_1} \subset G$. □

Theorem 3.4. *Let G be an unmixed graph with $2n$ vertices, which are not isolated, with* $\operatorname{height} I(G) = n$ *satisfying the condition (*). Then the following conditions are equivalent:*

(1) *G is Cohen-Macaulay.*
(2) *$\Delta(G)$ is strongly connected.*
(3) *The cycle C_{ij} is not included in G for any $i < j$.*

Proof. (1) $\Longrightarrow$ (2). Well known.

(2) $\Longrightarrow$ (3). Assume that $C_{ij} \subset G$ for some $i < j$. Let F be a facet of $\Delta(G)$ such that $x_i \in F$. Since $x_i y_j \in E(G)$, we have $y_j \notin F$ and by unmixedness of G it follows that $x_j \in F$. Hence $\{x_i, x_j\} \subset F$. Let F' be a facet of $\Delta(G)$ such that $\{y_i, y_j\} \subset F'$.

We show that there does not exist a chain of facets of $\Delta(G)$ such that

$$F = F_0, F_1, \dots, F_m = F', \text{ with } \sharp(F_i \cap F_{i+1}) = n - 1 \text{ for } i = 1, \dots, m - 1.$$

Every facet $H \in \Delta(G)$ is one of the following form:

$$H = \{z_1, \dots, z_{i-1}, x_i, z_{i+1}, \dots, z_{j-1}, x_j, z_{j+1}, \dots, z_n\}$$

or

$$H = \{z_1, \dots, z_{i-1}, y_i, z_{i+1}, \dots, z_{j-1}, y_j, z_{j+1}, \dots, z_n\},$$

where $z_k \in \{x_k, y_k\}$, since $\{x_i y_i, x_j y_j, x_i y_j, x_j y_i\} \subset E(G)$. Hence it is impossible to find such a chain. Hence $\Delta(G)$ is not strongly connected.

(3) $\Longrightarrow$ (1). In order to prove the statement by Proposition 3.2 it is sufficient to verify that $O_T(G)$ is unmixed for every subset T of $[n]$. Hence we prove that conditions (i) and (ii) of Proposition 2.3 are satisfied for the graph $G' = O_T(G)$.

First we check the condition (i) for G'. We may assume that $j \notin T$.

Suppose $i \notin T$. We must show the following:
"If $z_i x_j$, $y_j x_k \in E(G')$, then $z_i x_k \in E(G')$ for distict i, j and k and for $z_i \in \{x_i, y_i\}$."
Since $z_i x_j$, $y_j x_k \in E(G)$ and G is unmixed, by Theorem 2.3 we have $z_i x_k \in E(G)$. Hence $z_i x_k \in E(G')$.

Suppose $i \in T$. We must show the following:
"If $x_i x_j$, $y_j x_k \in E(G')$ then $x_i x_k \in E(G')$ for distict i, j and k."
Since $y_j x_k \in E(G')$, we have $y_j x_k \in E(G)$. Since $x_i x_j \in E(G')$, we have either $x_i x_j \in E(G)$ or $y_i x_j \in E(G)$. If $x_i x_j \in E(G)$, then by Theorem 2.3 we have $x_i x_k \in E(G)$, since $y_j x_k \in E(G)$ and G is unmixed. Similarly, if $y_i x_j \in E(G)$, then we have $y_i x_k \in E(G)$, since $y_j x_k \in E(G)$. In both cases, we have $x_i x_k \in E(G')$.

Next we check the condition (ii) for G'. We may assume that $j \notin T$. We also assume that $i \in T$. We must show that either $x_i x_j \notin E(G')$ or $x_i y_j \notin E(G')$. Suppose $x_i x_j, x_i y_j \in E(G')$. Then we have $x_i y_j \in E(G)$, and either $x_i x_j \in E(G)$ or $y_i x_j \in E(G)$. Since G is unmixed, $x_i x_j \in E(G)$

is impossible by Theorem 2.3, (ii). While the condition $y_ix_j \in E(G)$ is also impossible, since G does not have the cycle C_{ij} for any $i < j$. It is a contradiction. $\square$

The next lemma is crucial for giving another criterion for the Cohen-Macaulayness of our graphs.

Lemma 3.5. *Let G be an unmixed graph with $2n$ vertices, which are not isolated, and* height $I(G) = n$. *We assume the condition (*).*

If G is a Cohen-Macaulay graph then there exists a suitable simultaneous change of labeling on both $\{x_i\}$ and $\{y_i\}$ (i.e., we relable $(x_{i_1}, \dots, x_{i_n})$ and $(y_{i_1}, \dots, y_{i_n})$ as $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ at the same time), such that $x_iy_j \in E(G)$ implies $i \leq j$.

Proof. We can define a partial order $\preceq$ on X by

$$x_i \preceq x_j \text{ if and only if } x_iy_j \in E(G).$$

In fact, the reflexivity holds by (*), the transitivity holds by unmixedness of G (see Theorem 2.4 (i)) and the antisymmetry holds since G contains no cycle C_{ij} for any $i < j$. Take a linear extension of $\preceq$, which we call $\preceq'$. By the linear order $\preceq'$, we have $x_{i_1} \preceq' \cdots \preceq' x_{i_n}$. We relabel them as $x_1 \preceq' \cdots \preceq' x_n$. At the same time we relabel $y_{i_1}, \dots, y_{i_n}$ as $y_1, \dots, y_n$. Then if $x_iy_j \in E(G)$, $x_i \preceq' x_j$. Hence $i \leq j$. $\square$

Hence for a Cohen-Macaulay graph G with $2n$ vertices, which are not isolated, and height $I(G) = n$ satisfying the condition (*), we may assume that

(**) $x_iy_j \in E(G)$ implies $i \leq j$.

Now we state another Cohen-Macalay criterion on our graphs, which is generalization of Herzog-Hibi ([8], Theorem 3.4).

Theorem 3.6. *Let G be a graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$. *We assume the conditions (*) and (**).*

Then the following conditions are equivalent:

(1) *G is Cohen-Macaulay;*
(2) *G is unmixed;*
(3) *The following conditions hold:*
  (i) *If $z_ix_j, y_jx_k \in E(G)$ then $z_ix_k \in E(G)$ for distict i, j, k and for $z_i \in \{x_i, y_i\}$;*
  (ii) *If $x_iy_j \in E(G)$ then $x_ix_j \notin E(G)$.*

Proof. (1) $\Longrightarrow$ (2) is well known.
(2) $\Longrightarrow$ (1) follows from Theorem 3.4, since we assume the condition (**).
(2) $\Longleftrightarrow$ (3) follows from Theorem 2.3. $\square$

As an easy consequence of the previous results we obtain the upper bound for the minimal number $\mu(I(G))$ of generators of $I(G)$:

Corollary 3.7. *Let G be a graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$.

(i) *If G is unmixed, then $\mu(I(G)) \leq n^2$.*
(ii) *If G is Cohen-Macaulay, then $\mu(I(G)) \leq \frac{n(n+1)}{2}$.*

Proof. The statements are consequences of the criteria for the unmixedness and for the Cohen-Macaulayness given by Proposition 2.3 and Theorem 3.6. □

4. Shellability and Cohen-Macaulay type

In this section if G is a graph such that $\sharp V(G) = 2n$ and height $I(G) = n$, we show the equivalence between Cohen-Macaulayness of G and shellability of the complementary simplicial complex $\Delta(G)$. We also express the Cohen-Macaulay type of $S/I(G)$ in a combinatorial way.

Theorem 4.1. *Let G be an unmixed graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$. *Then G is Cohen-Macaulay if and only if $\Delta(G)$ is shellable.*

We just give a proof of the following lemma. The rest of the proof is almost identical with the proof of ([3], Theorem 2.9).

Lemma 4.2. *Let G be a Cohen-Macaulay graph with $2n$ vertices, which are not isolated, and* height $I(G) = n$. *Then there exists a vertex $v \in V(G)$ such that* $\deg(v) = 1$.

Proof. We may assume the condition (*). Suppose that each $v \in V(G)$ has at least degree 2. Let $i_1, i_2, \ldots$ be a sequence such that $y_{i_1}x_{i_2}, y_{i_2}x_{i_3}, \ldots \in E(G)$ with $i_j \neq i_{j+1}$. Since the cardinality of Y is finite, there must be exist integers $s < t$ such that $i_t = i_s$. We may assume that $i_s, i_{s+1}, \ldots, i_{t-1}$ are distinct. This induces the cycle $C_{i_s i_{s+1} \cdots i_{t-1}} \subset G$. Therefore G is not Cohen-Macaulay by Proposition 3.3 and Theorem 3.4. □

Now we express the Cohen-Macaulay type of a graph belonging to our class, imitating the bipartite case (see [14], pp. 184-185).

Lemma 4.3. *Let G be a Cohen-Macaulay graph with $2n$ vertices, which are not isolated, and* height $I(G) = n$. *We assume the condition (*). Then*

$$\mathrm{Soc}\ \left(K[x_1, \ldots, x_n] / \left(I(O_{[n]}(G)|_X) + (x_1^2, \ldots, x_n^2)\right)\right)$$

is generated by all the monomials $x_{i_1} \cdots x_{i_r}$ such that $\{x_{i_1}, \ldots, x_{i_r}\}$ is a maximal independent set of $O_{[n]}(G)|_X$.

Proof. The ring $A := K[x_1, \dots, x_n]/\left(I(O_{[n]}(G)|_X) + (x_1^2, \dots, x_n^2)\right)$ is spanned as a K-vector space by the image of 1 and the images of the squarefree monomials

$$x_{i_1} \cdots x_{i_r}, \qquad 1 \le i_1 < i_2 < \ldots < i_r \le n \tag{4.1}$$

such that $x_{i_j} x_{i_k} \notin E(O_{[n]}(G)|_X)$, for $j \neq k$, i.e. $\{x_{i_1}, \dots, x_{i_r}\}$ is an independent set of $O_{[n]}(G)|_X$. Since A is an artinian positively graded algebra, Soc $A = (0 :_A A_+)$ is generated by the images of the squarefree monomials of the form (4.1) such that $\{x_{i_1}, \dots, x_{i_r}\}$ is a maximal independent set of $O_{[n]}(G)|_X$. □

Corollary 4.4. *Let G be a Cohen-Macaulay graph with $2n$ vertices, which are not isolated, and* height $I(G) = n$. *We assume the condition (*). Then*

(i) type $S/I(G) = \sharp\Upsilon(O_{[n]}(G)|_X)$, *where $\Upsilon(O_{[n]}(G)|_X)$ is the family of all minimal vertex covers of $O_{[n]}(G)|_X$. In particular,* type $S/I(G)$ *is independent from the base field K.*

(ii) *G is level if and only if $O_{[n]}(G)|_X$ is unmixed. In particular, levelness of G is independent from the base field K.*

Proof. Set $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ and $S_n = K[x_1, \dots, x_n]$.

(i)Since G is Cohen-Macaulay and $\{x_1 - y_1, \dots, x_n - y_n\}$ is a regular sequence, we have

$$\begin{aligned} \text{type } S/I(G) &= \dim_K \operatorname{Soc} S/\left(I(G) + (x_1 - y_1, \dots, x_n - y_n)\right) \\ &= \dim_K \operatorname{Soc} S_n/\left(I(O_{[n]}(G)|_X) + (x_1^2, \dots, x_n^2)\right) \\ &= \sharp\Upsilon(O_{[n]}(G)|_X) \end{aligned}$$

by the previous lemma.

(ii) When G is Cohen-Macaulay, G is level if and only if

$$\operatorname{Soc} S/\left(I(G) + (x_1 - y_1, \dots, x_n - y_n)\right)$$

is equi-generated. By the previous lemma it is equivalent to that $O_{[n]}(G)|_X$ is unmixed. □

Corollary 4.5. *Let G be a Cohen-Macaulay graph with $2n$ vertices, which are not isolated, and* height $I(G) = n$. *We assume the condition (*). Then the following conditions are equivalent:*

(1) *G is Gorestein;*
(2) *$I(G) = (x_1y_1, \dots, x_ny_n)$;*
(3) *G is a complete intersection.*

Proof. (1) $\Rightarrow$ (2). G is Gorenstein if and only if $S/I(G)$ is Cohen-Macaulay and type $S/I(G) = 1$. Since $1 = \text{type } S/I(G) = \sharp\Upsilon(O_{[n]}(G)|_X)$, it follows that $O_{[n]}(G)|_X$ has a unique minimal vertex cover. Hence $O_{[n]}(G)|_X$ is isolated n vertices. Hence $I(G) = (x_1y_1, \dots, x_ny_n)$.

(2) $\Rightarrow$ (3). From its definition.
(3) $\Rightarrow$ (1). See [1]. $\square$

5. B-grafted graph

In this section we introduce a new class of graphs G with $\sharp V(G) = 2n$ and with height $I(G) = n$ and we study its Cohen-Macaulayness.

Let H_0 be a graph with the labeled vertices $1, 2, \ldots, p$.

For every $i = 1, \ldots, p$ let B_i be a bipartite graph with labeled partition X_i and Y_i such that $\sharp X_i = \sharp Y_i = n_i$. (We do not give a label to each vertex of B_i, but we distinguish the partition X_i and Y_i.) We assume that B_i has no isolated vertex for every $i = 1, \ldots, p$. We define the graph

$$G = G(H_0; B_1, \ldots, B_p)$$

as follows: The vertex set of G is $V(G) := X \cup Y$, where $X = X_1 \cup \ldots \cup X_p$, and $Y = Y_1 \cup \ldots \cup Y_p$. The edge set $E(G)$ of G defined by:

$$xy \in E(G) \text{ if and only if}$$

either

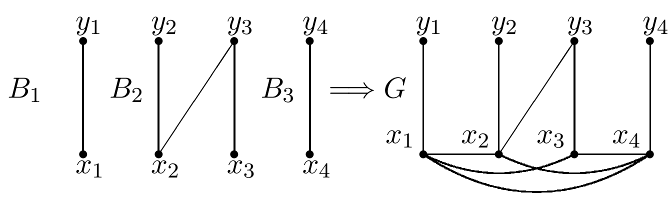
$$\text{there exist } i, j \quad \text{such that} \quad x \in X_i, y \in X_j, \text{ and } ij \in E(H_0)$$

or

$$\text{there exists } i \quad \text{such that} \quad x \in X_i, y \in Y_i, \text{ and } xy \in E(B_i).$$

We call such a graph G the *B-grafted graph*. Note that X is a minimal vertex cover of G and Y is a maximal independent set of G. Note also that $\sharp V(G) = 2(\sum_{i=1}^{p} n_i)$.

Example 5.1. Let H_0 be a cycle of the length 3. By the following bipartite graphs B_1, B_2, B_3, we obtain the B-grafted graph G:



Remark 5.2. If B_i is just a complete graph with 2 vertices, i.e., a complete bipartite graph with $\sharp X_i = \sharp Y_i = 1$ for $i = 1, \ldots, p$, then the B-grafted graph G is called a grafted graph in [4].

Theorem 5.3. *The B-grafted graph $G(H_0; B_1, \ldots, B_p)$ is Cohen-Macaulay (unmixed, respectively) if and only if every bipartite graph B_i is Cohen-Macaulay (unmixed, respectively) for $i = 1, \ldots, p$.*

Proof. It is clear from Theorem 3.4 (Proposition 2.3, respectively). $\square$

Acknowledgments. The third author acknowledges the financial support of GNSAGA-INDAM and the hospitality during his stay at the Department of Mathematics of the University of Messina (Italy). This work was also supported by KAKENHI18540041 and KAKENHI20540047.

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